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BY
WOLFGANG M. SCHMIDT

ABSTRACT. If $x_1, x_2, \dots$ is a sequence in the unit interval $0 \le x < 1$ and if S is a subinterval, write $C(n,S)$ for the number of elements among $x_1, \dots, x_n$ which lie in S, minus n times the length of S. For a well distributed sequence, $C(n,S)$ as a function of n will be small. It is shown that the lengths of the intervals S for which $C(n,S)\,(n = 1,2,\dots)$ is bounded form at most a countable set.

1. Introduction. The present paper is independent of the preceding papers of this series. However, the reader would be advised to first read the sixth paper [3] of the series, which deals with a similar but rather simpler problem.

We shall be concerned with the distribution of an arbitrary given sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$ of points in the unit cube of k-dimensional Euclidean space. This unit cube $\mathbf{U}^k$ consists of the points $\mathbf{x} = (x_1, \dots, x_k)$ with $0 \le x_i < 1\,(i = 1, \dots, k)$.

Let $\mathbf{S}$ be a measurable subset of $\mathbf{U}^k$ of measure $\mu(\mathbf{S})$, and write $Z(n, \mathbf{S})$ for the number of points among $\mathbf{x}_1, \dots, \mathbf{x}_n$ which lie in $\mathbf{S}$. The quantity

$$D(n,\mathbf{S}) = |Z(n,\mathbf{S}) - n\mu(\mathbf{S})| \tag{1}$$

tells us how far $Z(n,\mathbf{S})$ deviates from the "expected" number $n\mu(\mathbf{S})$. Put

$$E(\mathbf{S}) = \sup_n D(n,\mathbf{S}), \tag{2}$$

and call $E(\mathbf{S})$ the *error* of $\mathbf{S}$. We shall show in the present paper that very few boxes $\mathbf{B}$ with sides parallel to the coordinate axes have a finite error $E(\mathbf{B})$.

By a *subinterval* of $\mathbf{U}^1$ we shall mean a single point or an open, half-open or closed interval of positive length which is contained in $\mathbf{U}^1$. By a *box* contained in $\mathbf{U}^k$ we shall understand a set $\mathbf{B} = \mathbf{I}_1 \times \cdots \times \mathbf{I}_k$, where $\mathbf{I}_1, \cdots, \mathbf{I}_k$ are subintervals of $\mathbf{U}^1$. Thus $\mathbf{B}$ consists of points $\mathbf{x} = (x_1, \dots, x_k)$ with $x_j \in \mathbf{I}_j\,(j = 1, \dots, k)$.(2) Write $\mathfrak{B}_t$ for the class of sets which are unions of at most t boxes in $\mathbf{U}^k$.

For $\kappa > 0$, let $M_t(\kappa)$ be the set of numbers μ of the type $\mu = \mu(\mathbf{A})$ where $\mathbf{A} \in \mathfrak{B}_t$ and $E(\mathbf{A}) < \kappa$. Let $M_t(\infty)$ be the set of numbers μ of the type $\mu = \mu(\mathbf{A})$ with $\mathbf{A} \in \mathfrak{B}_t$ and $E(\mathbf{A}) < \infty$; thus $M_t(\infty)$ is the union of the sets $M_t(\kappa)$ with $0 < \kappa < \infty$. Recall that a number γ is a *limit point* of a set M of reals if there is a sequence of distinct elements of M which converge to γ. The *derivative* $M^{(1)}$ of

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(2) In the theory of uniform distribution, one usually studies the "discrepancy" function $D(n) = \sup D(n,\mathbf{B})$, with the supremum taken over all boxes $\mathbf{B}$ in $\mathbf{U}^k$, rather than the error $E(\mathbf{B})$, as in the present paper.

M consists of all the limit points of M. The higher derivatives are defined inductively by $M^{(d)} = (M^{(d-1)})^{(1)}$ $(d = 2, 3, \ldots)$.

Theorem 1. *Suppose $d > 8\kappa > 0$. Then $(M_t(\kappa))^{(d)}$ is empty.*

Since a set M having $M^{(d)}$ empty for some d is nowhere dense and is at most countable, we obtain the

Corollary. *Each set $M_t(\kappa)$ is nowhere dense and is at most countable. The set $M_t(\infty)$ is at most countable.*

In [3] we proved a result like Theorem 1 in the one-dimensional case for intervals $\mathbf{I}$ whose left endpoint was 0. The generalization to arbitrary intervals causes considerable additional difficulties in the proof; the generalization to arbitrary dimension and the generalization to unions of t boxes are easy.

As far as I know, the only interesting sequences for which the boxes $\mathbf{B}$ with finite $E(\mathbf{B})$ have been determined are the sequences

$$x_n = \{\alpha n + \beta\} \qquad (n = 1, 2, \ldots) \tag{3}$$

in the one-dimensional case, where the notation $\{\xi\}$ denotes the fractional part of ξ. Let us define an *interval modulo* 1 as either a subinterval $\mathbf{I}$ of $\mathbf{U}^1$ or the union of two subintervals $\mathbf{I}_1$, $\mathbf{I}_2$ of $\mathbf{U}^1$ such that $0 \in \mathbf{I}_1$ and $\mathbf{I}_2$ contains every number less than 1 which is sufficiently close to 1. (In particular, every interval modulo 1 lies in $\mathfrak{B}_2$. A suitably defined *box modulo* 1 would lie in $\mathfrak{B}_{2k}$; in fact we defined the class $\mathfrak{B}_t$ in order to allow boxes modulo 1.) Now for a sequence (3), $E(\mathbf{J})$ is finite where $\mathbf{J}$ is an interval modulo 1 if (Ostrowski [2]) and only if (Kesten [1]; this is the hard part) $\mu(\mathbf{J}) = \{\alpha l\}$ for some integer l. In particular, the set $M_1(\infty)$ is infinite if α is irrational.

No particular importance attaches to the number 8κ in Theorem 1. But in [3] it was shown that the van der Corput sequence has $(M_1(d))^{(d)}$ nonempty for $d = 1, 2, \ldots$.

Theorem 1 probably remains true if the class $\mathfrak{B}_t$ is replaced by the class $\mathfrak{P}_t$ of polyhedrons with at most t faces or by the class $\mathfrak{E}$ of ellipsoids contained in $\mathbf{U}^k$. But Theorem 1 is not true for the class of convex subsets of $\mathbf{U}^k$ when $k > 1$:

Theorem 2. *Suppose $k > 1$. There is a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbf{U}^k$ such that for every μ in $0 \leq \mu \leq 1$, there is a convex set $\mathbf{S}$ in $\mathbf{U}^k$ with $\mu(\mathbf{S}) = \mu$ and with $E(\mathbf{S}) \leq \frac{1}{2}$.*

The fairly easy proof of Theorem 2 is given in the last section and is independent of the rest of the paper. The proof of Theorem 1 is unfortunately rather long. It would be desirable to have a simpler proof.

2. Preliminaries. Every interval is of one of the following four types: (i) $\alpha \leq x \leq \beta$, (ii) $\alpha \leq x < \beta$, (iii) $\alpha < x \leq \beta$, (iv) $\alpha < x < \beta$. Now for a box $\mathbf{B} = \mathbf{I}_1 \times \cdots \times \mathbf{I}_k$, each interval $\mathbf{I}_j$ is of one of four possible types, and hence we have 4^k types of boxes. There are 4^{kt} types of t-tuples of boxes $\mathbf{B}_1, \ldots, \mathbf{B}_t$, and hence there is a finite number of types of elements of $\mathfrak{B}_t$. It clearly will suffice to prove Theorem 1 for each subclass of $\mathfrak{B}_t$ whose elements are of a given type. For the sake of simplicity we shall only deal with the type where each interval **I** used in the definition of boxes is of the type $\alpha \le x < \beta$. Denote such an interval by $\mathbf{I}[\alpha, \beta)$.

For $1 \le i \le k$, let $\mathbf{B}_i(\alpha)$ be the set of points $\mathbf{x}$ in $\mathbf{U}^k$ with $\alpha \le x_i < 1$, and for $k + 1 \le i \le 2k$, let $\mathbf{B}_i(\alpha)$ be the box of points $\mathbf{x}$ in $\mathbf{U}^k$ with $0 \le x_{i-k} < \alpha$. For $\alpha_1, \dots, \alpha_{2k}$ with $0 \le \alpha_j \le 1 \, (j = 1, \dots, 2k)$, put

$$\mathbf{B}(\alpha_1, \dots, \alpha_{2k}) = \mathbf{B}(\alpha_1) \cap \cdots \cap \mathbf{B}(\alpha_{2k}).$$

Then $\mathbf{B}(\alpha_1, \dots, \alpha_{2k})$ is a box, and every box of the type described above may be written as $\mathbf{I}[\alpha_1, \alpha_{k+1}) \times \cdots \times \mathbf{I}[\alpha_k, \alpha_{2k}) = \mathbf{B}(\alpha_1, \dots, \alpha_{2k})$. For $\alpha_1, \dots, \alpha_{2kt}$ with $0 \le \alpha_j \le 1 \, (j = 1, \dots, 2kt)$, put

$$\mathbf{A}(\alpha_1, \dots, \alpha_{2kt}) = \mathbf{B}(\alpha_1, \dots, \alpha_{2k}) \cup \mathbf{B}(\alpha_{2k+1}, \dots, \alpha_{4k}) \cup \cdots \cup \mathbf{B}(\alpha_{2k(t-1)+1}, \dots, \alpha_{2kt}).$$

Then $\mathbf{A}(\alpha_1, \dots, \alpha_{2kt})$ is in $\mathfrak{B}_t$, and every element of $\mathfrak{B}_t$ of the type described above is a set $\mathbf{A}(\alpha_1, \dots, \alpha_{2kt})$.

It will be convenient to write $2kt = m$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{2kt})$ and $\mathbf{A}(\boldsymbol{\alpha}) = \mathbf{A}(\alpha_1, \dots, \alpha_{2kt})$. Also put $\mu(\boldsymbol{\alpha}) = \mu(\mathbf{A}(\boldsymbol{\alpha}))$. The vectors $\boldsymbol{\alpha}$ will be restricted to the closed cube $\mathbf{C}$ in R^m defined by $0 \le \alpha_i \le 1 \, (i = 1, \dots, m)$.

We shall call a finite or infinite sequence of real numbers $\alpha^{(1)}, \alpha^{(2)}, \dots$ *monotonic* of the type $<$ if $\alpha^{(1)} < \alpha^{(2)} < \dots$, monotonic of the type $=$ if $\alpha^{(1)} = \alpha^{(2)} = \dots$, and monotonic of the type $>$ if $\alpha^{(1)} > \alpha^{(2)} > \cdots$. Every infinite sequence of real numbers contains an infinite monotonic subsequence. A finite or infinite sequence of vectors $\boldsymbol{\alpha}^{(1)} = (\alpha_1^{(1)}, \dots, \alpha_m^{(1)})$, $\boldsymbol{\alpha}^{(2)} = (\alpha_1^{(2)}, \dots, \alpha_m^{(2)})$, $\dots$ will be called monotonic of the type $(u_1, \dots, u_m)$ where each u_h is either $<$ or $=$ or $>$, if for $1 \le h \le m$, the sequence $\alpha_h^{(1)}, \alpha_h^{(2)}, \dots$ is monotonic of the type u_h.

Given subsets $\mathbf{A}$, $\mathbf{A}'$ of $\mathbf{U}^k$, the *symmetric difference*

$$\mathbf{A} \,\hat{}\, \mathbf{A}'$$

is the set of elements $\mathbf{x}$ which lie in $\mathbf{A}$ but not in $\mathbf{A}'$, or in $\mathbf{A}'$ but not in $\mathbf{A}$.

Lemma 1. *Suppose* $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \dots$ *is a monotonic sequence of vectors in* $\mathbf{C}$. *Then no point* $\mathbf{x}$ *of* $\mathbf{U}^k$ *lies in more than* $m = 2kt$ *of the sets* $\mathbf{A}(\boldsymbol{\alpha}^{(1)}) \,\hat{}\, \mathbf{A}(\boldsymbol{\alpha}^{(2)}), \mathbf{A}(\boldsymbol{\alpha}^{(2)}) \,\hat{}\, \mathbf{A}(\boldsymbol{\alpha}^{(3)}), \dots$.

Proof. $\mathbf{A}(\boldsymbol{\alpha}) = \mathbf{A}(\alpha_1, \dots, \alpha_{2kt})$ is formed as a union and intersection of the $2kt$ sets $\mathbf{B}_i(\alpha_{i+2kj}) \, (1 \le i \le 2k, 0 \le j \le t-1)$. Therefore if for every i, j with $1 \le i \le 2k$, $0 \le j \le t - 1$, the point $\mathbf{x}$ behaves the same way with respect to $\mathbf{B}_i(\alpha_{i+2kj})$ and $\mathbf{B}_i(\alpha'_{i+2kj})$, i.e. lies in both or in neither of them, then $\mathbf{x}$ lies in either both $\mathbf{A}(\boldsymbol{\alpha})$ and $\mathbf{A}(\boldsymbol{\alpha}')$ or in neither of them. Hence if $\mathbf{x} \in \mathbf{A}(\boldsymbol{\alpha}) \,\hat{}\, \mathbf{A}(\boldsymbol{\alpha}')$, then there are i, j with $1 \le i \le 2k$, $0 \le j \le t - 1$, such that $\mathbf{x} \in \mathbf{B}_i(\alpha_{i+2kj}) \,\hat{}\, \mathbf{B}_i(\alpha'_{i+2kj})$.

Thus to prove Lemma 1, it will suffice to show that for fixed i, j, a point $\mathbf{x}$ lies in at most one of the sets

$$(4) \qquad \mathbf{B}_i(\alpha_{i+2kj}^{(1)}) \hat{}\ \mathbf{B}_i(\alpha_{i+2kj}^{(2)}), \qquad \mathbf{B}_i(\alpha_{i+2kj}^{(2)}) \hat{}\ \mathbf{B}_i(\alpha_{i+2kj}^{(3)}), \qquad \ldots .$$

But $\mathbf{B}_i(\alpha)$ decreases with α if $1 \leq i \leq k$, and it increases with α if $k + 1 \leq i \leq 2k$. The sequence $\alpha_{i+2kj}{}^{(1)}, \alpha_{i+2kj}{}^{(2)}, \ldots$ is monotonic. Hence the sequence of sets $\mathbf{B}_i(\alpha_{i+2kj}{}^{(1)}), \mathbf{B}_i(\alpha_{i+2kj}{}^{(2)}), \ldots$ is either increasing or decreasing (in the weak sense). Therefore $\mathbf{x}$ can lie in at most one of the sets (4).

The property enunciated in Lemma 1, together with the continuity of $\mu(\boldsymbol{\alpha})$, are the only properties of the sets $\mathbf{A}(\boldsymbol{\alpha})$ which we shall need. It would be easy to construct other parameter families of sets with these properties, and hence other families of sets for which Theorem 1 holds.

3. **Directed systems.** Let $(i_1, \ldots, i_d)$ and $(i'_1, \ldots, i'_d)$ be d-tuples of positive integers. We shall write

$$(i_1, \ldots, i_d) \underset{j}{<} (i'_1, \ldots, i'_d)$$

if $i_1 = i'_1, \ldots, i_{j-1} = i'_{j-1}$ and $i_j < i'_j$.

We are going to define *directed systems* of real numbers. A directed system of order 0 consists of a single real number α in $0 \leq \alpha \leq 1$. A directed system of order 1 is a finite monotonic sequence of reals $\alpha(1), \alpha(2), \ldots, \alpha(l)$ with $l > 1$ terms in $0 \leq \alpha \leq 1$. Thus a directed system of order 1 is of some type (u) where u may be $<$, $=$ or $>$. For arbitrary $d \geq 1$, a directed system of order d is of some type $(u_1, \ldots, u_d)$, where each u_i may be $<$, $=$ or $>$, and consists of integers $l_1, \ldots, l_d$ greater than 1 and of real numbers $\alpha(i_1, \ldots, i_d)$ $(1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d)$ in the interval $0 \leq \alpha \leq 1$, such that if $1 \leq i_1, i'_1 \leq l_1, \ldots, 1 \leq i_d, i'_d \leq l_d$ and $(i_1, \ldots, i_d) <_j (i'_1, \ldots, i'_d)$, then

$$\alpha(i_1, \ldots, i_d)\, u_j\, \alpha(i'_1, \ldots, i'_d).$$

For example, in a directed system of the type $(<, \ldots, <)$, the numbers $\alpha(i_1, \ldots, i_d)$ are ordered lexicographically.

Lemma 2. *Suppose there exists a directed system of the type $(u_1, \ldots, u_{j-1}, =, u_{j+1}, \ldots, u_d)$ where $j < d$. Then the symbols $u_{j+1}, \ldots, u_d$ are all the $=$ sign, i.e. the type is $(u_1, \ldots, u_{j-1}, =, \ldots, =)$.*

Proof. Let $\alpha(i_1, \ldots, i_d)$ belong to a directed system of the type $(u_1, \ldots, u_{j-1}, =, u_{j+1}, \ldots, u_d)$. Then $\alpha(1, \ldots, 1, 1, i_{j+1}, \ldots, i_d) = \alpha(1, \ldots, 1, 2, i'_{j+1}, \ldots, i'_d)$ for any numbers $1 \leq i_{j+1}, i'_{j+1} \leq l_{j+1}, \ldots, 1 \leq i_d, i'_d \leq l_d$. Hence the $l_{j+1} \ldots l_d$ numbers $\alpha(1, \ldots, 1, i_{j+1}, \ldots, i_d)$ with $1 \leq i_{j+1} \leq l_{j+1}, \ldots, 1 \leq i_d \leq l_d$ are all equal, and the symbols $u_{j+1}, \ldots, u_d$ must be the $=$ sign.

Next, we define directed systems of vectors $\boldsymbol{\alpha}$. A directed system of order zero consists of a single vector $\boldsymbol{\alpha}$ in the cube $\mathbf{C}$. A directed system of order d where $d \geq 1$ is of some type $(u_{ih})(1 \leq i \leq d, 1 \leq h \leq m)$, where each u_{ih} is either $<$

or = or >, and consists of integers $l_1, \dots, l_d$ greater than 1 and of vectors $\alpha(i_1, \dots, i_d)$ $(1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d)$ such that for each h in $1 \leq h \leq m$, the coordinates $\alpha_h(i_1, \dots, i_d)(1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d)$ form a directed system of reals of the type $(u_{1h}, \dots, u_{dh})$. That is, we have

$$\alpha_h(i_1, \dots, i_d)\, u_{jh}\, \alpha_h(i'_1, \dots, i'_d)$$

if $1 \leq i_1, i'_1 \leq l_1, \dots, 1 \leq i_d, i'_d \leq l_d$ and $(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$.

4. A proposition which implies Theorem 1. By a *range* we shall understand a finite set of consecutive positive integers. Thus a range N will consist of integers $a+1, a+2, \dots, b$ where $0 \leq a < b$. The number $|N| = b - a$ will be called the *length* of the range, so that a range of length l contains exactly l integers. Now let $f(n)$ be a function defined on the positive integers, and let N be a range. We put

$$f^+(N) = \max_{n \in N} f(n), \qquad f^-(N) = \min_{n \in N} f(n), \tag{5}$$

and

$$f^*(N) = f^+(N) - f^-(N). \tag{6}$$

For $\boldsymbol{\alpha}$ in $\mathbf{C}$, write

$$f(n, \boldsymbol{\alpha}) = n\mu(\boldsymbol{\alpha}) - Z(n, \mathbf{A}(\boldsymbol{\alpha})). \tag{7}$$

The meaning of the notations $f^+(N, \boldsymbol{\alpha})$, $f^-(N, \boldsymbol{\alpha})$ and $f^*(N, \boldsymbol{\alpha})$ is then obvious.

Given a subset $\mathbf{S}$ of the cube $\mathbf{C}$, write $M(\mathbf{S})$ for the set of numbers $\mu = \mu(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in \mathbf{S}$. For any set M of real numbers, put $M^{(0)} = M$.

Proposition.(3) *Suppose $d \geq 0$ and $\mathbf{S}$ is a subset of $\mathbf{C}$ such that $(M(\mathbf{S}))^{(d)}$ contains a number μ in $0 < \mu < 1$. Suppose $\varepsilon > 0$. Then there is*

(i) *a positive integer r,*

(ii) *a directed system of vectors $\boldsymbol{\alpha}(i_1, \dots, i_d)(1 \leq i_j \leq l_j)$ of order d with $\boldsymbol{\alpha}(i_1, \dots, i_d) \in \mathbf{S}$ and $0 < \mu(\boldsymbol{\alpha}(i_1, \dots, i_d)) < 1$, and*

(iii) *there are neighborhoods*(4) $\mathbf{N}(i_1, \dots, i_d)$ *of the numbers $\mu(\boldsymbol{\alpha}(i_1, \dots, i_d))$, with the following property:*

If N is a range with $|N| \geq r$, and if

$$\boldsymbol{\beta}(i_1, \dots, i_d) \qquad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d)$$

is a directed system with $\mu(\boldsymbol{\beta}(i_1, \dots, i_d)) \in \mathbf{N}(i_1, \dots, i_d)$, but not necessarily of the same type as the directed system $\boldsymbol{\alpha}(i_1, \dots, i_d)$, then

$$(l_1 \dots l_d)^{-1} \sum_{i_1=1}^{l_1} \cdots \sum_{i_d=1}^{l_d} f^*(N, \boldsymbol{\beta}(i_1, \dots, i_d)) \geq \frac{1}{4}(d+1) + \frac{1}{12} - \varepsilon. \tag{8}$$

(3) This proposition corresponds to the proposition in [3]. Also, Lemmas 3, 7 of the present paper correspond, respectively, to Lemmas 5, 4 of [3].

(4) By a *neighborhood* of a real number μ we understand an open interval containing μ.

It might be in order to clarify the meaning of the proposition when $d = 0$. In this case the directed system consists of a single vector $\boldsymbol{\alpha} \in \mathbf{S}$. The hypothesis implies only that there is an $\boldsymbol{\alpha} \in \mathbf{S}$ with $0 < \mu(\boldsymbol{\alpha}) < 1$. Hence this case may be restated as follows.

Case $d = 0$ of the proposition. *Suppose $\boldsymbol{\alpha} \in \mathbf{C}$ with $0 < \mu(\boldsymbol{\alpha}) < 1$, and suppose $\varepsilon > 0$. Then there exists an integer r and a neighborhood $\mathbf{N}$ of $\mu(\boldsymbol{\alpha})$ such that for every range N with $|N| \geq r$ and every $\boldsymbol{\beta}$ with $\mu(\boldsymbol{\beta}) \in \mathbf{N}$, we have*

$$f^*(N,\boldsymbol{\beta}) \geq \frac{1}{4} + \frac{1}{12} - \varepsilon.$$

The proof of the proposition will be postponed until later. At present we are going to show that the proposition implies Theorem 1. We have to show that $(M_t(\kappa))^{(d)}$ is empty if $d > 8\kappa > 0$. It will suffice to show that $(M_t(\kappa))^{(d)}$ contains no element μ with $0 < \mu < 1$ if $d > 8\kappa - 1$. Put differently, it will be enough to show that if $(M_t(\kappa))^{(d)}$ contains an element μ with $0 < \mu < 1$, then

$$\kappa \geq \tfrac{1}{8}(d + 1). \tag{9}$$

By what we said in §2, we may restrict ourselves to sets of $\mathfrak{B}_t$ of the type $\mathbf{A}(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in \mathbf{C}$. Thus if $\overline{M}_t(\kappa)$ is the set of numbers $\mu = \mu(\boldsymbol{\alpha})$ with $E(\mathbf{A}(\boldsymbol{\alpha})) < \kappa$, we have to show that (9) holds if there is a $\mu \in (\overline{M}_t(\kappa))^{(d)}$ with $0 < \mu < 1$. Let $\mathbf{S}(\kappa)$ consist of the vectors $\boldsymbol{\alpha}$ with $E(\mathbf{A}(\boldsymbol{\alpha})) < \kappa$. Then $\overline{M}_t(\kappa) = M(\mathbf{S}(\kappa))$. If $(\overline{M}_t(\kappa))^{(d)} = (M(\mathbf{S}(\kappa)))^{(d)}$ contains an element μ with $0 < \mu < 1$, we apply the proposition with $\varepsilon = 1/12$, with a range N having $|N| \geq r$, and with $\boldsymbol{\beta}(i_1, \ldots, i_d) = \boldsymbol{\alpha}(i_1, \ldots, i_d)$ (or with $\boldsymbol{\beta} = \boldsymbol{\alpha}$ if $d = 0$), and we see that there is a $\boldsymbol{\beta} \in \mathbf{S}(\kappa)$ with

$$f^*(N,\boldsymbol{\beta}) \geq \tfrac{1}{4}(d + 1).$$

Hence either $|f^+(N,\boldsymbol{\beta})| \geq \frac{1}{8}(d + 1)$ or $|f^-(N,\boldsymbol{\beta})| \geq \frac{1}{8}(d + 1)$, and there is an integer $n \in N$ with $|f(n,\boldsymbol{\beta})| \geq \frac{1}{8}(d + 1)$. Thus

$$D(n, \mathbf{A}(\boldsymbol{\beta})) = |Z(n, \mathbf{A}(\boldsymbol{\beta})) - n\mu(\boldsymbol{\beta})| = |f(n,\boldsymbol{\beta})| \geq \tfrac{1}{8}(d + 1),$$

and $E(\mathbf{A}(\boldsymbol{\beta})) \geq \frac{1}{8}(d + 1)$. Since $\boldsymbol{\beta} \in \mathbf{S}(\kappa)$ and $E(\mathbf{A}(\boldsymbol{\beta})) < \kappa$, we obtain (9).

5. **An auxiliary lemma.** If $f(n)$ is a function defined on the positive integers, and if N, N' are ranges, put

$$f^\nabla(N,N') = \max(0, f^-(N) - f^+(N'), f^-(N') - f^+(N)). \tag{10}$$

Lemma 3. *Let $f(n)$, $g(n)$ be defined on the positive integers, and let L, L' be subranges of a range N. Then*

$$f^*(N) + g^*(N) \geq (f - g)^\nabla(L,L') + \tfrac{1}{2}(f^*(L) + g^*(L) + f^*(L') + g^*(L')).$$

Proof. Since both L, L' are contained in N, we have $f^*(N) \geq \max(f^*(L),$

$f^*(L'))$ and $g^*(N) \geq \max(g^*(L), g^*(L'))$, so that the lemma is certainly true if $(f-g)^\nabla(L,L') = 0$. We therefore may assume without loss of generality that

$$(f-g)^\nabla(L,L') = (f-g)^-(L) - (f-g)^+(L') > 0.$$

Then we have for every $l \in L$ and every $l' \in L'$,

$$f(l) - g(l) - (f(l') - g(l')) \geq (f-g)^\nabla(L,L'). \tag{11}$$

Let a_f, b_f, a_g, b_g be integers in L with

$$f(a_f) = f^+(L), \qquad f(b_f) = f^-(L),$$
$$g(a_g) = g^+(L), \qquad g(b_g) = g^-(L),$$

so that

$$f(a_f) - f(b_f) = f^*(L), \tag{12}$$

$$g(a_g) - g(b_g) = g^*(L). \tag{13}$$

Similarly, choose a'_f, b'_f, a'_g, b'_g in L' with

$$f(a'_f) - f(b'_f) = f^*(L'), \tag{14}$$

$$g(a'_g) - g(b'_g) = g^*(L'). \tag{15}$$

Applying (11) with $l = a_g$, $l' = a'_f$, we get

$$f(a_g) - g(a_g) - f(a'_f) + g(a'_f) \geq (f-g)^\nabla(L,L').$$

Applying (11) with $l = b_f$, $l' = b'_g$, we obtain

$$f(b_f) - g(b_f) - f(b'_g) + g(b'_g) \geq (f-g)^\nabla(L,L').$$

Adding these two inequalities and the four equations (12), (13), (14), (15), we obtain

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geq 2(f-g)^\nabla(L,L') + f^*(L) + g^*(L) + f^*(L') + g^*(L'),$$

where

$$\varphi_1 = f(a_f) - f(b'_f), \qquad \varphi_2 = g(a'_g) - g(b_g),$$
$$\varphi_3 = f(a_g) - f(b'_g), \qquad \varphi_4 = g(a'_f) - g(b_f).$$

Since $f^*(N) \geq \max(\varphi_1, \varphi_3)$ and $g^*(N) \geq \max(\varphi_2, \varphi_4)$, the lemma follows.

The lemma will not be used until §11.

6. The case $d = 0$ of the proposition. Write $\|\xi\|$ for the distance from a real number ξ to the nearest integer. Suppose $\mu = \mu(\alpha)$ lies in the open interval $0 < \mu < 1$. Then there is a positive integer q such that $\|\mu q\| \geq \frac{1}{3}$. This follows from Kronecker's theorem if μ is irrational, and is easily proved if μ is rational, the worst case being when $\mu = \frac{1}{3}$ or $\frac{2}{3}$. Choose a neighborhood $\mathbf{N}$ of μ such that $|\mu' - \mu| < \varepsilon/q$ for every $\mu' \in \mathbf{N}$. Then for every $\boldsymbol{\beta}$ with $\mu(\boldsymbol{\beta}) \in \mathbf{N}$,

$$\|\mu(\boldsymbol{\beta})q\| > \tfrac{1}{3} - \varepsilon.$$

Now put $r = q + 1$, and let $N = \{a + 1, a + 2, \ldots, b\}$ be a range of length $|N| \geq r$. Then there are two integers n, n' in N, e.g. $n = a + 1$ and $n' = a + 1 + q$, such that

$$\|n\mu(\boldsymbol{\beta}) - n'\mu(\boldsymbol{\beta})\| > \tfrac{1}{3} - \varepsilon.$$

Hence

$$|f(n', \boldsymbol{\beta}) - f(n, \boldsymbol{\beta})| \geq \|n\mu(\boldsymbol{\beta}) - n'\mu(\boldsymbol{\beta})\| > \tfrac{1}{3} - \varepsilon,$$

and in the notation of (6) we have

$$f^*(N, \boldsymbol{\beta}) > \frac{1}{3} - \varepsilon = \frac{1}{4} + \frac{1}{12} - \varepsilon.$$

This finishes the proof of the case $d = 0$ of the proposition. The proposition in general will later be proved by induction on d.

7. Kronecker type lemmas.

Lemma 4. *There are positive valued functions $f_1(y_0), f_2(y_0, y_1), f_3(y_0, y_1, y_2), \ldots,$ defined for nonzero $y_0, y_1, y_2, \ldots,$ with the following property:*

Suppose l is a positive integer and

$$\begin{aligned} &0 < \varepsilon < 1, \quad 0 < |\delta_1| < f_1(\varepsilon), \quad 0 < |\delta_2| < f_2(\varepsilon, \delta_1), \quad \ldots, \\ &\qquad\qquad\qquad\qquad 0 < |\delta_l| < f_l(\varepsilon, \delta_1, \ldots, \delta_{l-1}). \end{aligned} \tag{16}$$

Then there is a positive integer $p = p(l, \varepsilon, \delta_1, \ldots, \delta_l)$ such that, for every range N with $|N| \geq p$ and arbitrary $\alpha_1, \ldots, \alpha_l$, there is an $n \in N$ with

$$\{\delta_i n - \alpha_i\} < \varepsilon \qquad (i = 1, \ldots, l). \tag{17}$$

Proof. Put $f_1(y_0) = |y_0|$. Suppose $0 < \varepsilon < 1$ and $0 < |\delta_1| < f_1(\varepsilon) = \varepsilon$. Suppose at first that $\delta_1 > 0$, and put $p = [1/\delta_1] + 1$, where $[\xi]$ denotes the integer part of a real number ξ. The numbers $z_0 = 0, z_1 = \delta_1, \ldots, z_{p-1} = [1/\delta_1]\delta_1$ lie in $0 \leq z \leq 1$, and given any α there is a z_i with $\{z_i - \alpha\} < \delta_1 < \varepsilon$. Thus there is an n with $0 \leq n \leq p - 1$ and $\{\delta_1 n - \alpha\} < \varepsilon$. Since this holds for *every* α, it is easily seen that in every range N with $|N| \geq p$, there is an n with $\{\delta_1 n - \alpha\} < \varepsilon$. The situation is similar if $\delta_1 < 0$.

Now suppose $l \geq 2$ and $f_1, \ldots, f_{l-1}$ have been constructed and have the desired properties. Suppose

(18) $0 < \varepsilon < 1, \quad 0 < |\delta_1| < f_1(\varepsilon), \quad \ldots, \quad 0 < |\delta_{l-1}| < f_{l-1}(\varepsilon, \delta_1, \ldots, \delta_{l-2}).$

Put $p' = p(l-1, \varepsilon, \delta_1, \ldots, \delta_{l-1})$ and

$$f_l(\varepsilon, \delta_1, \ldots, \delta_{l-1}) = \varepsilon/2p'. \tag{19}$$

(Clearly, it does not matter how we define f_l if (18) is violated.)

Suppose

$$0 < |\delta_l| < f_l(\varepsilon, \delta_1, \ldots, \delta_{l-1}). \tag{20}$$

Then $0 < |\delta_l| < \frac{1}{2}\varepsilon = f_1(\frac{1}{2}\varepsilon)$, and by the case $l = 1$ of the lemma there is a $p'' = p(1, \frac{1}{2}\varepsilon, \delta_l)$ such that for every range N'' with $|N''| \geq p''$ and every α_l, there is an $n'' \in N''$ with $\{\delta_l n'' - \alpha_l\} < \frac{1}{2}\varepsilon$. Put

$$p = p(l, \varepsilon, \delta_1, \ldots, \delta_l) = p' + p''.$$

Now let $\alpha_1, \ldots, \alpha_l$ be arbitrary, and let N be a range with $|N| \geq p$. Assume at first that $\delta_l > 0$, and let N'' be the subrange of N with $|N''| = p''$ and with its smallest element the same as that of N. There is an $n'' \in N''$ with $\{\delta_l n'' - \alpha_l\} < \frac{1}{2}\varepsilon$. Let N' be the range $n'', n'' + 1, \ldots, n'' + p' - 1$, so that $N' \subseteq N$. There is an $n \in N'$ with

$$\{\delta_i n - \alpha_i\} < \varepsilon \qquad (i = 1, \ldots, l-1).$$

Furthermore, $\{\delta_l n - \alpha_l\} = \{\delta_l(n - n'') + \delta_l n'' - \alpha_l\} = \delta_l(n - n'') + \{\delta_l n'' - \alpha_l\} < \delta_l p' + \frac{1}{2}\varepsilon < \varepsilon$ by (19) and (20). Hence (17) holds for $i = 1, \ldots, l$.

The situation is analogous if $\delta_l < 0$. In this case we let the largest element of N'' coincide with that of N.

Lemma 5. *Suppose l is a positive integer and suppose $\varepsilon, \delta_1, \ldots, \delta_l$ satisfy* (16). *Let $p = p(l, \varepsilon, \delta_1, \ldots, \delta_l)$ be the number p of Lemma* 4. *There are neighborhoods $\mathbf{D}_1$ of $\delta_1, \ldots, \mathbf{D}_l$ of δ_1 (which may depend on $l, \varepsilon, \delta_1, \ldots, \delta_l$) with the following property*:

Suppose N, N' are ranges with lengths $|N| \geq p$, $|N'| \geq p$. Suppose $\eta_1 \in \mathbf{D}_1, \ldots, \eta_l \in \mathbf{D}_l$. Then there are integers $n \in N$, $n' \in N'$ such that

$$\|n\eta_i - n'\eta_i\| > \tfrac{1}{2} - \varepsilon \qquad (i = 1, \ldots, l). \tag{21}$$

Proof. Choose the neighborhood $\mathbf{D}_i$ of δ_i so small that

$$p|\eta_i - \delta_i| < \tfrac{1}{2}\varepsilon \tag{22}$$

for every $\eta_i \in \mathbf{D}_i (i = 1, \ldots, l)$. Now if N, N' are ranges with $|N| \geq p$, $|N'| \geq p$, pick n' arbitrary in N' and let n_0 be the smallest element in N.

Suppose $\eta_1 \in \mathbf{D}_1, \ldots, \eta_l \in \mathbf{D}_l$ are given. Put

$$\alpha_i = (n' - n_0)\eta_i + n_0\delta_i + \tfrac{1}{2} - \tfrac{1}{2}\varepsilon \qquad (i = 1, \ldots, l).$$

By Lemma 4 there is an $n \in N$ with $n_0 \leq n \leq n_0 + p - 1$ and with $\{n\delta_i - \alpha_i\} < \varepsilon (i = 1, \ldots, l)$. This is the same as

$$\tfrac{1}{2} - \tfrac{1}{2}\varepsilon \leq \{n\delta_i - (n' - n_0)\eta_i - n_0\delta_i\} < \tfrac{1}{2} + \tfrac{1}{2}\varepsilon.$$

Now

$$n\eta_i - n'\eta_i = n\delta_i - (n' - n_0)\eta_i - n_0\delta_i + (n - n_0)(\eta_i - \delta_i),$$

and since $|n - n_0| < p$ and since η_i in $\mathbf{D}_i$ satisfies (22) and hence $|n - n_0|\,|\eta_i - \delta_i| < \frac{1}{2}\varepsilon$, we obtain $\frac{1}{2} - \varepsilon < \{n\eta_i - n'\eta_i\} < \frac{1}{2} + \varepsilon$, which is equivalent to (21).

We shall say that a function $g(n)$ *is of the type* η, where η is a real number, if

$$\{g(n+1) - g(n) - \eta\} = 0 \qquad (n = 1, 2, \ldots).$$

If N, N' are ranges of positive integers, we put

$$g^{\square}(N, N') = \min_{n \in N} \min_{n' \in N'} |g(n) - g(n')|. \tag{23}$$

This is not to be confused with the notation $f^{\nabla}(N, N')$ of (10).

Lemma 6. *Suppose l, ε, $\delta_1, \ldots, \delta_l$ and p and $\mathbf{D}_1, \ldots, \mathbf{D}_l$ are as in Lemma 5. Let r be a positive integer, and assume that $\delta_1, \ldots, \delta_l$ satisfy the inequalities*

$$0 < |\delta_i| < \varepsilon/(2r) \qquad (i = 1, \ldots, l), \tag{24}$$

and in fact that every $\eta_i \in \mathbf{D}_i$ satisfies

$$|\eta_i| < \varepsilon/(2r). \tag{25}$$

Then there is an integer $p^* = p^*(l, \varepsilon, \delta_1, \ldots, \delta_l; r)$, such that if $\eta_1 \in \mathbf{D}_1, \ldots, \eta_l \in \mathbf{D}_l$, and if $g_1(n), \ldots, g_l(n)$ are functions of the types $\eta_1, \ldots, \eta_l$, respectively, and if N, N' are ranges with $|N| \geq p^*$, $|N'| \geq p^*$, then there are subranges $L \subseteq N$, $L' \subseteq N'$ with

$$|L| = |L'| = r \tag{26}$$

and with

$$g_i^{\square}(L, L') > \tfrac{1}{2} - 2\varepsilon \qquad (i = 1, \ldots, l). \tag{27}$$

Proof. Put $p^* = \max(p, r)$. Suppose $\eta_1 \in \mathbf{D}_1, \ldots, \eta_l \in \mathbf{D}_l$ and $|N| \geq p^*$, $|N'| \geq p^*$. By Lemma 5, there are integers $n \in N$ and $n' \in N'$ with (21). There is a range $L \subseteq N$ with $n \in L$ and $|L| = r$, and a range $L' \subseteq N'$ with $n' \in L'$ and $|L'| = r$. For $m \in L$ and $m' \in L'$ we have $|(m - n)\eta_i| \leq r|\eta_i| < \frac{1}{2}\varepsilon$ and $|(m' - n')\eta_i| < \frac{1}{2}\varepsilon$, so that, by (21),

$$\begin{aligned}|g_i(m) - g_i(m')| &\geq \|g_i(m) - g_i(m')\| = \|m\eta_i - m'\eta_i\| \\ &= \|n\eta_i - n'\eta_i + (m - n)\eta_i - (m' - n')\eta_i\| > \tfrac{1}{2} - \varepsilon - 2\varepsilon/2 > \tfrac{1}{2} - 2\varepsilon.\end{aligned}$$

Thus (27) holds.

8. **Functions** $f(n,\alpha,\beta)$. For α, β in **C**, put

$$f(n,\alpha,\beta) = f(n,\alpha) - f(n,\beta) = n(\mu(\alpha) - \mu(\beta)) - (Z(n,\mathbf{A}(\alpha)) - Z(n,\mathbf{A}(\beta))). \tag{28}$$

The function $f(n,\alpha,\beta)$ is of the type $\eta = \mu(\alpha) - \mu(\beta)$.

Lemma 7. *Suppose $\alpha_1, \alpha_2, \ldots$ are elements of* **C** *such that the numbers $\mu(\alpha_1), \mu(\alpha_2), \ldots$ are all distinct and converge to a number μ with $0 < \mu < 1$. Suppose $0 < \varepsilon < 1$, $l \geq 1$, $r \geq 1$.*

There is a finite subsequence $\alpha(1), \ldots, \alpha(2l)$ of $\alpha_1, \alpha_2, \ldots$ with $0 < \mu(\alpha(j)) < 1$ $(j = 1, \ldots, 2l)$, and there are neighborhoods $\mathbf{N}_j$ of $\mu(\alpha(j))$, and there is an integer q, with the following properties:

For any $\beta(1), \ldots, \beta(2l)$ with $\mu(\beta(j)) \in \mathbf{N}_j$ $(j = 1, \ldots, 2l)$, we have

$$|\mu(\beta(2i-1)) - \mu(\beta(2i))| < \varepsilon/(4r) \qquad (i = 1, \ldots, l). \tag{29}$$

Furthermore, if N, N' are ranges with $|N| \geq q$, $|N'| \geq q$, there are subranges $L \subseteq N$, $L' \subseteq N'$ with

$$|L| = |L'| = r \tag{30}$$

and with

$$f^{\square}(L, L', \beta(2i-1), \beta(2i)) > \tfrac{1}{2} - \varepsilon \qquad (i = 1, \ldots, l). \tag{31}$$

Proof. There is an integer j_1 with

$$0 < |\mu(\alpha_{j_1}) - \mu(\alpha_{j_1+1})| < \min(f_1(\tfrac{1}{2}\varepsilon), \varepsilon/(4r)),$$

where f_1 is the function of Lemma 4. Put $\delta_1 = \mu(\alpha_{j_1}) - \mu(\alpha_{j_1+1})$. There is a $j_2 > j_1 + 1$ such that

$$0 < |\mu(\alpha_{j_2}) - \mu(\alpha_{j_2+1})| < \min(f_2(\tfrac{1}{2}\varepsilon, \delta_1), \varepsilon/(4r)).$$

Put $\delta_2 = \mu(\alpha_{j_2}) - \mu(\alpha_{j_2+1})$. We continue in this manner, and choose integers $j_1 < j_1 + 1 < j_2 < j_2 + 1 < \ldots < j_l < j_l + 1$, such that the numbers

$$\delta_i = \mu(\alpha_{j_i}) - \mu(\alpha_{j_i+1}) \qquad (i = 1, \ldots, l)$$

satisfy (16) and (24), with ε replaced by $\varepsilon/2$. Let $\mathbf{D}_i = \mathbf{D}_i(l, \varepsilon/2, \delta_1, \ldots, \delta_l)$ $(i = 1, \ldots, l)$ and $p^* = p^*(l, \varepsilon/2, \delta_1, \ldots, \delta_l; r)$ be as in Lemma 6. Put

$$\alpha(1) = \alpha_{j_1}, \quad \alpha(2) = \alpha_{j_1+1}, \quad \cdots, \quad \alpha(2l-1) = \alpha_{j_l}, \quad \alpha(2l) = \alpha_{j_l+1},$$

so that $\mu(\alpha(2i-1)) - \mu(\alpha(2i)) = \delta_i$ $(i = 1, \ldots, l)$. Choose neighborhoods $\mathbf{N}_j$ of $\mu(\alpha(j))$ $(j = 1, \ldots, 2l)$ so small that $\mu_{2i-1} - \mu_{2i} \in \mathbf{D}_i$ if $\mu_{2i-1} \in \mathbf{N}_{2i-1}$ and μ_{2i}

$\in \mathbf{N}_{2l}$. Then if $\boldsymbol{\beta}(1), \ldots, \boldsymbol{\beta}(2l)$ are vectors with $\mu(\boldsymbol{\beta}(j)) \in \mathbf{N}_j (j = 1, \ldots, 2l)$, (29) follows from the condition (25) (but with $\varepsilon/2$ in place of ε) on the neighborhoods $\mathbf{D}_1, \ldots, \mathbf{D}_l$.

Now suppose that $|N| \geq q$, $|N'| \geq q$ where $q = p^*(l, \varepsilon/2, \delta_1, \ldots, \delta_l; r)$. The function $g_i(n) = f(n, \boldsymbol{\beta}(2i-1), \boldsymbol{\beta}(2i))$ is of the type $\mu(\boldsymbol{\beta}(2i-1)) - \mu(\boldsymbol{\beta}(2i)) = \eta_i$, say, where $\eta_i \in \mathbf{D}_i (i = 1, \ldots, l)$. By Lemma 6, there are ranges $L \subseteq N$, $L' \subseteq N'$ with (26) and (27) (but with ε replaced by $\varepsilon/2$), i.e. with (30) and (31). This finishes the proof of Lemma 7.

Suppose again that $g(n)$ is a function of the type η. We shall say that g *has a jump at* n if either $n = 1$ and $g(1) \neq \eta$, or if $n > 1$ and $g(n) - g(n-1) - \eta \neq 0$. Recall that the function $f(n, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is defined in terms of a given sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$.

Lemma 8. *The function $f(n, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of the type $\mu(\boldsymbol{\alpha}) - \mu(\boldsymbol{\beta})$ has a jump at n if and only if*

$$\mathbf{x}_n \in \mathbf{A}(\boldsymbol{\alpha})\,\hat{}\,\mathbf{A}(\boldsymbol{\beta}).$$

Proof. $f(n, \boldsymbol{\alpha}, \boldsymbol{\beta})$ has a jump at n precisely if

$$Z(n, \mathbf{A}(\boldsymbol{\beta})) - Z(n, \mathbf{A}(\boldsymbol{\alpha})) - (Z(n-1, \mathbf{A}(\boldsymbol{\beta})) - Z(n-1, \mathbf{A}(\boldsymbol{\alpha}))) \neq 0.$$

(Here we set $Z(0, \mathbf{A}) = 0$.) This holds if and only if $\mathbf{x}_n \in \mathbf{A}(\boldsymbol{\alpha})\,\hat{}\,\mathbf{A}(\boldsymbol{\beta})$.

9. **The construction of directed systems.** Suppose the numbers $\alpha(i_1, \ldots, i_d)$ with $1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d$ belong to a directed system of the type $(u_1, \ldots, u_d)$. Let u be $<$ or $=$ or $>$. Put

$$\begin{aligned}(u_1^*, \ldots, u_d^*, u_{d+1}^*) &= (u_1, \ldots, u_d, u) && \text{if } u_d \text{ is not } =, \\ &= (u_1, \ldots, u_k, u, \ldots, u) && \text{if for some } k < d, \text{ the symbols}\end{aligned}$$
$$u_{k+1}, \ldots, u_d \text{ are } =, \text{ and either } k = 0 \text{ or } u_k \text{ is not } = .$$

We shall describe a process of constructing a directed system of order $d + 1$ of the type $(u_1^*, \ldots, u_d^*, u_{d+1}^*)$. Let K be the set of d-tuples $(i_1, \ldots, i_d)$ with $1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d$. A subset H of K will be called a *segment* if $(i_1, \ldots, i_d) \in H$ whenever $(i_1', \ldots, i_d') \in H$ and $(i_1, \ldots, i_d) <_j (i_1', \ldots, i_d')$ for some j. Let l_{d+1} be an integer greater than 1. A *partial directed system* of order $d + 1$ on H will mean a system of numbers $\alpha(i_1, \ldots, i_d, i_{d+1})$ defined for $(i_1, \ldots, i_d) \in H$ and $1 \leq i_{d+1} \leq l_{d+1}$, such that

(a) $\alpha(i_1, \ldots, i_d, i_{d+1})\, u_j^*\, \alpha(i_1', \ldots, i_d', i_{d+1}')$ if $(i_1, \ldots, i_d) \in H$, $(i_1', \ldots, i_d') \in H$ and $(i_1, \ldots, i_d, i_{d+1}) <_j (i_1', \ldots, i_d', i_{d+1}')$ for some j in $1 \leq j \leq d + 1$.

(b) $\alpha(i_1, \ldots, i_d, i_{d+1})\, u_j^*\, \alpha(i_1', \ldots, i_d')$ if $(i_1, \ldots, i_d) \in H$, $(i_1', \ldots, i_d') \notin H$ and $(i_1, \ldots, i_d) <_j (i_1', \ldots, i_d')$ for some j in $1 \leq j \leq d$.

In particular, if H is empty, the empty set is a partial system defined on H. If $H = K$, a partial directed system on H is a directed system of order $d + 1$.

Lemma 9. *Suppose H is a segment, and H^* the segment which consists of H and a single further d-tuple $(t_1, \ldots, t_d)$. Suppose*

$$\alpha(i_1, \ldots, i_d, i_{d+1}) \qquad ((i_1, \ldots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$$

is a partial directed system defined on H.(5) *Suppose there is a sequence $\alpha_s(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$ which is monotonic of the type u and tends to $\alpha(t_1, \ldots, t_d)$. Then if s_0 is sufficiently large, and if we put*

$$\alpha(t_1, \ldots, t_d, i) = \alpha_{s_0+i}(t_1, \ldots, t_d) \qquad (1 \leq i \leq l_{d+1}), \tag{32}$$

then the numbers

$$\alpha(i_1, \ldots, i_d, i_{d+1}) \qquad ((i_1, \ldots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1})$$

are a partial directed system defined on H^*.

Proof. The condition (a) is satisfied if $(i_1, \ldots, i_d), (i'_1, \ldots, i'_d) \in H$. In order that it also be satisfied for H^*, we have to satisfy the following two conditions.

(a_1^*) $\qquad \alpha(i_1, \ldots, i_d, i_{d+1})\, u_j^*\, \alpha(t_1, \ldots, t_d, i'_{d+1})$ if $(i_1, \ldots, i_d) \underset{j}{<} (t_1, \ldots, t_d)$

for some j in $1 \leq j \leq d$.

(a_2^*) $\qquad \alpha(t_1, \ldots, t_d, i_{d+1})\, u_{d+1}^*\, \alpha(t_1, \ldots, t_d, i'_{d+1})$ if $i_{d+1} < i'_{d+1}$.

Since $(t_1, \ldots, t_d)$ is the only new element in H^*, (b) will be satisfied for H^* if

(b^*) $\alpha(t_1, \ldots, t_d, i_{d+1})\, u_j^*\, \alpha(i'_1, \ldots, i'_d)$ whenever $(t_1, \ldots, t_d) \underset{j}{<} (i'_1, \ldots, i'_d)$.

Now since $\alpha_s(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$ is monotonic of the type $u = u_{d+1}^*$, the condition (a_2^*) will be satisfied if $\alpha(t_1, \ldots, t_d, i)$ is given by (32), no matter how we choose s_0.

Now suppose that $1 \leq j \leq d$ and u_j^* is not $=$. Suppose $(i_1, \ldots, i_d) <_j (t_1, \ldots, t_d)$. Since $(i_1, \ldots, i_d)$ is in H and $(t_1, \ldots, t_d)$ is not in H, we have $\alpha(i_1, \ldots, i_d, i_{d+1})\, u_j^*\, \alpha(t_1, \ldots, t_d)$ by the hypothesis (b). Since $\alpha_s(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$ tends to $\alpha(t_1, \ldots, t_d)$, (a_1^*) will be satisfied if s_0 in (32) is sufficiently large. Next, suppose that $1 \leq j \leq d$ and u_j^* is $=$. Then also u_j and u are $=$. Suppose $(i_1, \ldots, i_d) <_j (t_1, \ldots, t_d)$. Then $\alpha(i_1, \ldots, i_d, i_{d+1}) = \alpha(t_1, \ldots, t_d)$ by (b). Since $\alpha_s(t_1, \ldots, t_d) = \alpha(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$, (a_1^*) will certainly be true. Thus (a_1^*) can always be satisfied.

Now suppose that $1 \leq j \leq d$ and u_j is not $=$. Then u_j^* equals u_j and is not $=$. Since $\alpha(i_1, \ldots, i_d)$ is a directed system of the type $(u_1, \ldots, u_d)$, we have $\alpha(t_1, \ldots, t_d)\, u_j^*\, \alpha(i'_1, \ldots, i'_d)$ if $(t_1, \ldots, t_d) <_j (i'_1, \ldots, i'_d)$. Since $\alpha_s(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$ tends to $\alpha(t_1, \ldots, t_d)$, (b^*) will be true if s_0 in (32) is sufficiently large. Finally, suppose that $1 \leq j \leq d$ and u_j is $=$. Then u_j^* is u. Since $\alpha(i_1, \ldots, i_d)$ is

(5) This part of the hypothesis does not apply if H is empty. Obvious changes have to be made in the proof in this case.

a directed system of the type $(u_1, \ldots, u_d)$, we have $\alpha(t_1, \ldots, t_d) = \alpha(i'_1, \ldots, i'_d)$ if $(t_1, \ldots, t_d) <_j (i'_1, \ldots, i'_d)$. Since $\alpha_s(t_1, \ldots, t_d)$ is monotonic of the type u and tends to $\alpha(t_1, \ldots, t_d)$, we have $\alpha(t_1, \ldots, t_\alpha, i_{d+1})\, u\, \alpha(t_1, \ldots, t_d) = \alpha(i'_1, \ldots, i'_d)$, i.e. $\alpha(t_1, \ldots, t_d, i_{d+1})\, u_j^*\, \alpha(i'_1, \ldots, i'_d)$. Thus (b*) will be satisfied.

This finishes the proof of Lemma 9. It is clear that by using an inductive argument and using Lemma 9 at each step, we can gradually build up a directed system of order $d+1$ of the type $(u_1^*, \ldots, u_{d+1}^*)$.

Now suppose that $\alpha(i_1, \ldots, i_d)$ with $1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d$ is a directed system of vectors of the type $(u_{ih})(1 \leq i \leq d, 1 \leq h \leq m)$. Let $u_1, \ldots, u_m$ be symbols $<$, $=$ or $>$. For each h in $1 \leq h \leq m$, put

$$\begin{aligned}(u_{1h}^*, \ldots, u_{dh}^*, u_{d+1,h}^*) &= (u_{1h}, \ldots, u_{dh}, u_h) && \text{if } u_{dh} \text{ is not } =,\\ &= (u_{1h}, \ldots, u_{k_h,h}, u_h, \ldots, u_h) && \text{if } u_{k_h+1,h}, \ldots, u_{dh} \text{ are } =,\\ & && \text{and if either } k_h = 0 \text{ or } u_{k_h,h} \text{ is not } = .\end{aligned}$$

We shall indicate a process to construct a directed system of vectors of order $d+1$ of the type $(u_{ih}^*)(1 \leq i \leq d+1, 1 \leq m \leq h)$. A *partial directed system of vectors* of order $d+1$ on a segment H is defined in the obvious way.

Lemma 10. *Suppose H is a segment and H^* is the segment consisting of H and of a single further d-tuple $(t_1, \ldots, t_d)$. Suppose*

$$\alpha(i_1, \ldots, i_d, i_{d+1}) \qquad ((i_1, \ldots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$$

is a partial directed system defined on H.(6) *Suppose there is a sequence of vectors $\alpha_s(t_1, \ldots, t_d)$ $(s = 1, 2, \ldots)$ which converges to $\alpha(t_1, \ldots, t_d)$, and which is monotonic of the type $(u_1, \ldots, u_m)$. Then if s_0 is sufficiently large, and if we put*

$$\alpha(t_1, \ldots, t_d, i) = \alpha_{s_0+i}(t_1, \ldots, t_d) \qquad (1 \leq i \leq l_{d+1}), \tag{33}$$

then the vectors

$$\alpha(i_1, \ldots, i_d, i_{d+1}) \qquad ((i_1, \ldots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1})$$

are a partial directed system defined on H^*.

Proof. We may use induction on the number m of components of our vectors, and use Lemma 9 at each step of the induction.

10. **Inductive proof of the proposition.** Let S be a subset of the cube C. A vector α will be called a *limit point* of S if there is a sequence $\alpha_1, \alpha_2, \ldots$ of elements of S which converge to α and which have distinct values $\mu(\alpha_1), \mu(\alpha_2), \ldots$. (This condition is more restrictive than the usual condition that $\alpha_1, \alpha_2, \ldots$ be distinct.) Let $S^{(1)}$ be the set of limit points of S. Since $\mu(\alpha)$ is a continuous function of α, it is clear that $M(S^{(1)}) \subseteq M^{(1)}(S)$. Conversely, if $\mu \in M^{(1)}(S)$, there are distinct elements $\mu_1 = \mu(\alpha_1), \mu_2 = \mu(\alpha_2), \ldots$ of $M(S)$ with $\alpha_1, \alpha_2, \ldots$ in S and with

(6) See the footnote to Lemma 9.

$\mu_i \to \mu$. There is a convergent subsequence of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$; let us denote the limit of this subsequence by $\boldsymbol{\alpha}$. Then $\boldsymbol{\alpha} \in \mathbf{S}^{(1)}$ and $\mu = \mu(\boldsymbol{\alpha})$, so that $\mu \in M(\mathbf{S}^{(1)})$. Hence

$$M^{(1)}(\mathbf{S}) = M(\mathbf{S}^{(1)}). \tag{34}$$

Every sequence $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$ has a subsequence which is monotonic of some type $(u_1, \ldots, u_m)$. If the numbers $\mu(\boldsymbol{\alpha}_1), \mu(\boldsymbol{\alpha}_2), \ldots$ are all distinct, this type cannot be $(=, \ldots, =)$. Hence if $\mathbf{S}^{(1)}(u_1, \ldots, u_m)$ consists of the elements $\boldsymbol{\alpha} \in \mathbf{S}^{(1)}$ for which there is a monotonic sequence $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$ in $\mathbf{S}$ of the type $(u_1, \ldots, u_m)$ which tends to $\boldsymbol{\alpha}$ and has distinct $\mu(\boldsymbol{\alpha}_1), \mu(\boldsymbol{\alpha}_2), \ldots$, then $\mathbf{S}^{(1)}$ is the union of the $3^m - 1$ sets $\mathbf{S}^{(1)}(u_1, \ldots, u_m)$ with $(u_1, \ldots, u_m)$ not $(=, \ldots, =)$.

Now assume that $d \geq 0$ and that $M^{(d+1)}(\mathbf{S})$ contains an element μ with $0 < \mu < 1$. By (34) we have $M^{(d+1)}(\mathbf{S}) = M^{(d)}(\mathbf{S}^{(1)})$, so that μ lies in one of the $3^m - 1$ sets $M^{(d)}(\mathbf{S}^{(1)}(u_1, \ldots, u_m))$ with $(u_1, \ldots, u_m)$ not $(=, \ldots, =)$. Suppose $(u_1, \ldots, u_m)$ is a particular m-tuple with

$$\mu \in M^{(d)}(\mathbf{S}^{(1)}(u_1, \ldots, u_m)).$$

We now assume the truth of the proposition for our particular value of d and apply it to $\mathbf{S}^{(1)}(u_1, \ldots, u_m)$. There is an integer $r = r^{(d)}$, a directed system $\boldsymbol{\alpha}(i_1, \ldots, i_d)$ with elements in $\mathbf{S}^{(1)}(u_1, \ldots u_m)$, and there are neighborhoods $\mathrm{N}(i_1, \ldots, i_d)$ with the properties enunciated in the proposition. Suppose this directed system

$$\boldsymbol{\alpha}(i_1, \ldots, i_d) \qquad (1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d)$$

is of the type $(u_{ih})(1 \leq i \leq d, 1 \leq h \leq m)$. Construct $(u^*_{ih})(1 \leq i \leq d+1, 1 \leq h \leq m)$ as in §9. The goal of the present section is a proof of the following

Lemma 11. *Suppose $l_{d+1} = 2l > 0$, $r > 0$, $\varepsilon > 0$. There is a directed system*

$$\boldsymbol{\alpha}(i_1, \ldots, i_d, i_{d+1}) \qquad (1 \leq i_1 \leq l_1, \ldots, 1 \leq i_{d+1} \leq l_{d+1})$$

of the type $(u^*_{ih})(1 \leq i \leq d+1, 1 \leq h \leq m)$, all of whose vectors $\boldsymbol{\alpha}$ lie in $\mathbf{S}$ and have $0 < \mu(\boldsymbol{\alpha}) < 1$, and there are neighborhoods $\mathrm{N}(i_1, \cdots, i_{d+1})$ of $\mu(\boldsymbol{\alpha}(i_1, \cdots, i_{d+1}))$, such that

$$\mathrm{N}(i_1, \ldots, i_d, i_{d+1}) \subseteq \mathrm{N}(i_1, \ldots, i_d) \qquad (1 \leq i_1 \leq l_1, \ldots, 1 \leq i_{d+1} \leq l_{d+1}). \tag{35}$$

Also, if $\boldsymbol{\beta} \in \mathrm{N}(i_1, \ldots, i_d, 2j-1)$, $\boldsymbol{\beta}' \in \mathrm{N}(i_1, \ldots i_d, 2j)$ with $1 \leq i_1 \leq l_1, \ldots, 1 \leq i_d \leq l_d$ and with $1 \leq j \leq l$, then

$$|\mu(\boldsymbol{\beta}) - \mu(\boldsymbol{\beta}')| < \varepsilon/4r. \tag{36}$$

Finally, there is an integer p such that if N, N' are ranges with $|N| \geq p$, $|N'| \geq p$, and if vectors $\boldsymbol{\beta}(i_1, \ldots, i_d, i_{d+1})$ have $\mu(\boldsymbol{\beta}(i_1, \ldots, i_{d+1})) \in \mathrm{N}(i_1, \ldots, i_{d+1})$ for $1 \leq i_1 \leq l_1, \ldots, 1 \leq i_{d+1} \leq l_{d+1}$, then there are subranges $L \subseteq N$, $L' \subsetneq N'$ with

$$|L| = |L'| = r \tag{37}$$

and with

$$f^{\square}(L, L', \beta(i_1, \ldots, i_d, 2j-1), \beta(i_1, \ldots, i_d, 2j)) > \tfrac{1}{2} - \varepsilon \qquad (1 \le i_1 \le l_1, \ldots, 1 \le i_d \le l_d, 1 \le j \le l). \tag{38}$$

We shall prove this lemma by constructing partial directed systems of order $d + 1$ defined on segments, and by gradually increasing these segments.

Lemma 12. *Suppose* $l_{d+1} = 2l > 0$, $v > 0$, $\varepsilon > 0$. *Let* H *be a nonempty segment. There is a partial directed system*

$$\alpha(i_1, \ldots, i_d, i_{d+1}) \qquad ((i_1, \ldots, i_d) \in H, 1 \le i_{d+1} \le l_{d+1})$$

of the type (u^*_{ih}) *with all the vectors* α *in* $\mathbf{S}$ *and satisfying* $0 < \mu(\alpha) < 1$, *and there are neighborhoods* $\mathrm{N}(i_1, \ldots, i_d, i_{d+1})$ *of* $\mu(\alpha(i_1, \ldots, i_d, i_{d+1}))$, *defined for* $(i_1, \ldots, i_d) \in H$, $1 \le i_{d+1} \le l_{d+1}$, *such that*

$$\mathrm{N}(i_1, \ldots, i_d, i_{d+1}) \subseteq \mathrm{N}(i_1, \ldots, i_d) \qquad ((i_1, \ldots, i_d) \in H, 1 \le i_{d+1} \le l_{d+1}), \tag{39}$$

and such that

$$|\mu(\beta) - \mu(\beta')| < \varepsilon/4v \tag{40}$$

if $\beta \in \mathrm{N}(i_1, \ldots, i_d, 2j-1)$, $\beta' \in \mathrm{N}(i_1, \ldots, i_d, 2j)$ *for some* $(i_1, \ldots, i_d) \in H$, $1 \le j \le l$. *There is an integer* $p = p(H)$ *such that if* N, N' *are ranges with* $|N| \ge p(H)$, $|N'| \ge p(H)$, *and if* $\beta(i_1, \ldots, i_d, i_{d+1}) \in \mathrm{N}(i_1, \ldots, i_d, i_{d+1})((i_1, \ldots, i_d) \in H, 1 \le i_{d+1} \le l_{d+1})$, *then there are subranges* $L \subseteq N$, $L' \subseteq N'$ *with*

$$|L| = |L'| = v \tag{41}$$

and with

$$f^{\square}(L, L', \beta(i_1, \ldots, i_d, 2j-1), \beta(i_1, \ldots, i_d, 2j)) > \tfrac{1}{2} - \varepsilon \qquad ((i_1, \ldots, i_d) \in H, 1 \le j \le l). \tag{42}$$

The case when $H = K$, the set of all d-tuples $(i_1, \ldots, i_d)$, is Lemma 11.

Proof of Lemma 12. We shall proceed by "induction on H". We shall assume that H is a segment properly contained in K, and that either H is empty, or H is nonempty and Lemma 12 is true for H. There is a unique segment H^* which consists of H and a single further d-tuple $(t_1, \ldots, t_d)$. We shall now prove Lemma 12 for H^*. We shall tacitly assume that H is nonempty; the necessary modifications in the argument when H is empty are trivial.

Suppose $\alpha(i_1, \ldots, i_d, i_{d+1})$ defined for $(i_1, \ldots, i_d) \in H$, $1 \le i_{d+1} \le l_{d+1}$, and respective neighborhoods $\mathrm{N}(i_1, \ldots, i_d, i_{d+1})$ and the number $p = p(H)$ have the desired properties as enunciated in Lemma 12 with respect to H.

Since $\alpha(t_1,\dots,t_d)$ lies in $S^{(l)}(u_1,\dots,u_m)$, there is a monotonic sequence $\alpha_s(t_1,\dots,t_d)(s = 1,2,\dots)$ of the type $(u_1,\dots,u_m)$ which tends to $\alpha(t_1,\dots,t_d)$, and is such that the numbers $\mu(\alpha_s(t_1,\dots,t_d))(s = 1,2,\dots)$ are all distinct. We now put $r = p(H) + v$ and apply Lemma 7 to l, r, ε and to the sequence $\alpha_1(t_1,\dots,t_d), \alpha_2(t_1,\dots,t_d), \dots$. There is a finite subsequence

$$\alpha(t_1,\dots,t_\alpha, 1), \dots, \alpha(t_1,\dots,t_d, 2l), \tag{43}$$

and there are neighborhoods $N(t_1,\dots,t_d,i)$ of $\mu(\alpha(t_1,\dots,t_d,i))(1 \le i \le 2l)$, and there is an integer q, with the following properties. For any $\beta(t_1,\dots,t_d,1)$, $\dots, \beta(t_1,\dots,t_d,2l)$ with $\mu(\beta(t_1,\dots,t_d,i)) \in N(t_1,\dots,t_d,i)(1 \le i \le 2l)$, we have

$$|\mu(\beta(t_1,\dots,t_d,2j-1)) - \mu(\beta(t_1,\dots,t_d,2j))| < \varepsilon/4r < \varepsilon/4v \qquad (1 \le j \le l). \tag{44}$$

Furthermore, if N, N' are ranges with $|N| \ge q$, $|N'| \ge q$, there are subranges $N_H \subseteq N$, $N'_H \subseteq N'$ with

$$|N_H| = |N'_H| = r > p(H) \tag{45}$$

and with

$$f^{\square}(N_H, N'_H, \beta(t_1,\dots,t_d,2j-1), \beta(t_1,\dots,t_d,2j)) > \tfrac{1}{2} - \varepsilon \qquad (1 \le j \le l). \tag{46}$$

Now the sequence $\alpha_s(t_1,\dots,t_d)(s = 1,2,\dots)$ may be replaced by a subsequence with $\mu(\alpha_s(\cdots))$ contained in $N(t_1,\cdots,t_d)$, and hence the sequence (43) can be chosen so that all its measures μ lie in $N(t_1,\cdots,t_d)$. The neighborhoods $N(t_1,\cdots,t_d,i)$ can be chosen so small that $N(t_1,\cdots,t_d,i) \subseteq N(t_1,\cdots,t_d)$ $(1 \leqslant i \leqslant 2l)$. This, together with the "inductive" assumption (39) yields

$$N(i_1,\dots,i_d,i_{d+1}) \subseteq N(i_1,\dots,i_d) \qquad ((i_1,\dots,i_d) \in H^*, 1 \le i_{d+1} \le l_{d+1}). \tag{47}$$

The "inductive" assumption (40) for H together with (44) gives a condition like (40) for H^*. Lemma 10 shows that, moreover, the sequence (43) can be chosen so that $\alpha(i_1,\dots,i_d,i_{d+1})$ with $(i_1,\dots,i_d) \in H^*$, $1 \le i_{d+1} \le l_{d+1}$, is a partial directed system on H^*.

Now suppose $\beta(i_1,\dots,i_d,i_{d+1}) \subseteq N(i_1,\dots,i_d,i_{d+1})((i_1,\dots,i_d) \in H^*, 1 \le i_{d+i} \le l_{d+1})$, $|N| \ge q$, $|N'| \ge q$, and suppose N_H, N'_H are chosen as above with (45) and (46). By the "inductive" assumption on H, there are subranges $L \subseteq N_H$, $L' \subseteq N'_H$ with (41), (42). Since $L \subseteq N_H$, $L' \subseteq N'_H$, the relations (42) together with (46) yield

$$f^{\square}(L, L', \beta(i_1,\dots,i_d,2j-1), \beta(i_1,\dots,i_d,2j)) > \tfrac{1}{2} - \varepsilon \qquad ((i_1,\dots,i_d) \in H^*, 1 \le i_{d+1} \le l_{d+1}).$$

Hence Lemma 12 is true for H^* with $p(H^*) = q$.

11. **Proof of the proposition completed.** We saw in §10 that if $M^{(d+1)}(\mathbf{S})$ contains an element μ in $0 < \mu < 1$, then $\mu \in M^{(d)}(\mathbf{S}^{(1)}(u_1, \ldots, u_m))$ for some m-tuple $(u_1, \ldots, u_m)$. We applied the inductive hypothesis to $\mathbf{S}^{(1)}(u_1, \ldots, u_m)$. There is a directed system $\boldsymbol{\alpha}(i_1, \ldots, i_d)$ of order d, and there are neighborhoods $\mathrm{N}(i_1, \ldots, i_d)$ and an integer $r = r^{(d)}$ with the properties stated in the proposition. Lemma 11 asserted the existence of a directed system $\boldsymbol{\alpha}(i_1, \ldots, i_d, i_{d+1})$ of order $d+1$, and neighborhoods $\mathrm{N}(i_1, \ldots, i_d, i_{d+1})$ and a number p which have certain properties in relation to the given directed system of order d.

Lemma 13. *Suppose we have the same hypotheses as in Lemma* 11, *and let* $\boldsymbol{\alpha}(i_1, \ldots, i_{d+1})$, $\mathrm{N}(i_1, \ldots, i_{d+1})$, p *be as in Lemma* 11. *Suppose*

$$(48) \qquad \boldsymbol{\beta}(i_1, \ldots, i_{d+1}) \qquad (1 \le i_1 \le l_1, \ldots, 1 \le i_{d+1} \le l_{d+1})$$

is a directed system with

$$(49) \quad \mu(\boldsymbol{\beta}(i_1, \ldots, i_{d+1})) \subseteq \mathrm{N}(i_1, \ldots, i_{d+1}) \qquad (1 \le i_1 \le l_1, \ldots, 1 \le i_{d+1} \le l_{d+1}),$$

but not necessarily of the same type as $\boldsymbol{\alpha}(i_1, \ldots, i_{d+1})$. *We know from Lemma* 11 *that if* $|N| \ge p$, $|N'| \ge p$, *then there are subranges* $L \subseteq N$, $L' \subseteq N'$ *with* (37) *and* (38). *We now claim that for every* $(i_1, \ldots, i_d)$ *with* $1 \le i_1 \le l_1, \ldots, 1 \le i_d \le l_d$, *we have*

$$(50) \quad \sum_{j=1}^{l} f^{\nabla}(L, L', \boldsymbol{\beta}(i_1, \ldots, i_d, 2j-1), \boldsymbol{\beta}(i_1, \ldots, i_d, 2j)) > l(\tfrac{1}{2} - 2\varepsilon) - 2mr.$$

Proof. Throughout, we keep $i_1, \ldots, i_d$ fixed. Since (48) is a directed system, the sequence $\boldsymbol{\beta}(i_1, \ldots, i_d, i)$ with $i = 1, 2, \ldots, 2l$ is monotonic. Hence, by Lemma 1, a point $\mathbf{x}_n$ can lie in at most m of the l sets

$$\mathrm{A}(\boldsymbol{\beta}(i_1, \ldots, i_d, 2j-1)\,\hat{}\;\mathrm{A}(\boldsymbol{\beta}(i_1, \ldots, i_d, 2j)) \qquad (j = 1, 2, \ldots, l).$$

It follows from Lemma 8 that at most m of the l functions

$$(51) \qquad f(n, \boldsymbol{\beta}(i_1, \ldots, i_d, 2j-1), \boldsymbol{\beta}(i_1, \ldots, i_d, 2j)) \qquad (j = 1, 2, \ldots, l)$$

can have a jump at n. Hence in view of (37), at most $2mr$ of these functions can have a jump at any $n \in L$ or any $n' \in L'$. It will suffice to show that if a function (51) has no jump in L or in L', then this function satisfies

$$(52) \qquad f^{\nabla}(L, L', \boldsymbol{\beta}(i_1, \ldots, i_d, 2j-1), \boldsymbol{\beta}(i_1, \ldots, i_d, 2j)) > \tfrac{1}{2} - 2\varepsilon.$$

Suppose $n_0 \in L$, $n_0' \in L'$. By (38), the values of our function (51) at $n = n_0$ and at $n = n_0'$ differ by at least $\frac{1}{2} - \varepsilon$. Without loss of generality we may assume that

$$f(n_0,\beta(i_1,\ldots,i_d,2j-1),\beta(i_1,\ldots,i_d,2j)) \\ -f(n_0',\beta(i_1,\ldots,i_d,2j-1),\beta(i_1,\ldots,i_d,2j)) > \tfrac{1}{2}-\varepsilon.$$

Now $f(n,\beta(i_1,\ldots,i_d,2j-1),\ \beta(i_1,\ldots,i_d,2j))$ is of the type $\mu(\beta(i_1,\ldots,i_d,2j-1)) - \mu(\beta(i_1,\ldots,i_d,2j))$ and has no jump in L. Hence for every $n \in L$,

$$\begin{aligned}&|f(n,\beta(\ldots,2j-1),\beta(\ldots,2j)) - f(n_0,\beta(\ldots,2j-1),\beta(\ldots,2j))| \\ &\quad = |n-n_0|\,|\mu(\beta(\ldots,2j-1)) - \mu(\beta(\ldots,2j))| \\ &\quad \le r(\varepsilon/4r) = \varepsilon/4,\end{aligned}$$

by virtue of (36) and (49). A similar inequality holds for every $n' \in L'$, and hence we have for every $n \in L$ and every $n' \in L'$,

$$f(n,\beta(\ldots,2j-1),\beta(\ldots,2j)) - f(n',\beta(\ldots,2j-1),\beta(\ldots,2j)) \\ > \tfrac{1}{2}-\varepsilon-2(\varepsilon/4) > \tfrac{1}{2}-2\varepsilon.$$

Therefore (52) holds, and Lemma 13 is true.

The proof of the proposition is now completed as follows. In Lemma 11 and in Lemma 13, the number l is still at our disposal. We now choose it so large that $l\varepsilon > 2mr$. Then the right-hand side of (50) may be replaced by $l(\frac{1}{2}-3\varepsilon)$. Since $f(n,\alpha,\beta) = f(n,\alpha) - f(n,\beta)$, we may rewrite (50) as

$$\sum_{j=1}^{l} (f(\ldots,\beta(i_1,\ldots,i_d,2j-1)) - f(\ldots,\beta(i_1,\ldots,i_d,2j)))^{\nabla}(L,L') \\ > l(\tfrac{1}{2}-3\varepsilon). \tag{53}$$

For every directed system $\beta(i_1,\ldots,i_{d+1})$ with (49) and every pair N, N' with $|N| \ge p$, $|N'| \ge p$, there are subranges $L \subsetneq N$, $L' \subsetneq N'$, with $|L| = |L'| = r$ and with (53) for arbitrary $i_1,\ldots,i_d$. In particular, in every single N with $|N| \ge p$, there are two subranges $L \subsetneq N$, $L' \subsetneq N'$ with $|L| = |L'| = r$ and with (53). By Lemma 3 applied to $f(n) = f(n,\beta(i_1,\ldots,i_d,2j-1))$ and $g(n) = f(n, \beta(i_1,\ldots,i_d,2j))$ we have

$$\begin{aligned}&f^*(N,\beta(i_1,\ldots,i_d,2j-1)) + f^*(N,\beta(i_1,\ldots,i_d,2j)) \\ &\quad\ge (f(\ldots,\beta(i_1,\ldots,i_d,2j-1)) - f(\ldots,\beta(i_1,\ldots,i_d,2j)))^{\nabla}(L,L') \\ &\qquad + \tfrac{1}{2}(f^*(L,\beta(i_1,\ldots,i_d,2j-1)) + f^*(L,\beta(i_1,\ldots,i_d,2j)) \\ &\qquad\quad + f^*(L',\beta(i_1,\ldots,i_d,2j-1)) + f^*(L',\beta(i_1,\ldots,i_d,2j))).\end{aligned}$$

We now take the sum over $j = 1, 2, \ldots, l$ and use (53) to obtain

$$\sum_{i_{d+1}=1}^{l_{d+1}} f^*(N,\beta(i_1,\ldots,i_d,i_{d+1})) > l\left(\frac{1}{2}-3\varepsilon\right) \\ + \frac{1}{2}\sum_{i_{d+1}=1}^{l_{d+1}} (f^*(L,\beta(i_1,\ldots,i_d,i_{d+1})) + f^*(L',\beta(i_1,\ldots,i_d,i_{d+1}))).$$

This holds for arbitrary $i_1, \ldots, i_d$ with $1 \le i_1 \le l_1, \ldots, 1 \le i_d \le l_d$. We now take the sum over all d-tuples $(i_1, \ldots, i_d)$ and divide by $l_1 \ldots l_d l_{d+1} = 2l_1 \ldots l_d l$. We obtain

$$(l_1 \ldots l_{d+1})^{-1} \sum_{i_1=1}^{l_1} \cdots \sum_{i_{d+1}=1}^{l_{d+1}} f^*(N, \beta(i_1, \ldots, i_{d+1})) > \frac{1}{4} - \frac{3}{2}\varepsilon$$
$$+ l_{d+1}^{-1} \sum_{i_{d+1}=1}^{l_{d+1}} \left(\frac{1}{2}(l_1 \ldots l_d)^{-1} \sum_{i_1=1}^{l_1} \cdots \sum_{i_d=1}^{l_d} (f^*(L, \beta(i_1, \ldots, i_{d+1})) + f^*(L', \beta(i_1, \ldots, i_{d+1}))) \right)$$
$$= \frac{1}{4} - \frac{3}{2}\varepsilon + l_{d+1}^{-1} \sum_{i_{d+1}=1}^{l_{d+1}} z(i_{d+1}),$$

say. Now for fixed i_{d+1}, $\beta(i_1, \ldots, i_d, i_{d+1})$ with $1 \le i_1 \le l_1, \ldots, 1 \le i_d \le l_d$, is a directed system of order d. We have

$$\mu(\beta(i_1, \ldots, i_d, i_{d+1})) \in \mathbf{N}(i_1, \ldots, i_d, i_{d+1}) \subseteq \mathbf{N}(i_1, \ldots, i_d)$$

by (35), (49), and both L, L' have length $r = r^{(d)}$. Hence by our inductive assumption, i.e. by the case d of the proposition, we have

$$z(i_{d+1}) \ge \frac{1}{4}(d+1) + \frac{1}{12} - \varepsilon,$$

whence

$$(54) \quad (l_1 \ldots l_{d+1})^{-1} \sum_{i_1=1}^{l_1} \cdots \sum_{i_{d+1}=1}^{l_{d+1}} f^*(N, \beta(i_1, \ldots, i_d, i_{d+1})) > \frac{1}{4}(d+2) + \frac{1}{12} - \frac{5}{2}\varepsilon.$$

This holds for every range N with $|N| \ge p = r^{(d+1)}(5\varepsilon/2)$, say. The whole construction could be carried out with $2\varepsilon/5$ in place of ε, and then our inequality (54) would become (8) with $d + 1$ in place of d.

This finishes our inductive proof of the proposition.

12. **Proof of Theorem 2.** It is easily seen that the general case of Theorem 2 follows from the 2-dimensional case, so that we may restrict ourselves to this case.

Pick numbers $t_1 > t_2 > \ldots$ with

$$(55) \qquad 0 < t_j < 1/(8j) \qquad (j = 1, 2, \ldots),$$

and let $\mathbf{x}_j = (x_j, y_j)$ be the point $(1 - \cos t_j, \sin t_j)$. Then the points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ lie on the circle $(x-1)^2 + y^2 = 1$, and they satisfy

$$(56) \qquad \sqrt{(x_j^2 + y_j^2)} < 1/(4j) \qquad (j = 1, 2, \ldots).$$

For every μ in $0 \le \mu \le 1$, we are going to construct sets $\mathbf{F}(n, \mu)$ $(n = 1, 2, \ldots)$ as follows. If $\mu < \frac{1}{2}$, let $\mathbf{F}(1, \mu)$ be empty, and if $\mu \ge \frac{1}{2}$, let $\mathbf{F}(1, \mu)$ consist of $\mathbf{x}_1$.

Then always

$$||\mathbf{F}(1,\mu)|-\mu| \leq \tfrac{1}{2}, \tag{57}$$

where $|\mathbf{F}|$ denotes the number of elements of a finite set $\mathbf{F}$. Now suppose $\mathbf{F}(n,\mu)$ has already been chosen and is a subset of $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ with

$$||\mathbf{F}(n,\mu)|-n\mu| \leq \tfrac{1}{2}. \tag{58}$$

Then $-(3/2) \leq |\mathbf{F}(n,\mu)| - (n+1)\mu \leq \frac{1}{2}$, so that either $||\mathbf{F}(n,\mu)|-(n+1)\mu| \leq \frac{1}{2}$ or $||\mathbf{F}(n,\mu)|+1-(n+1)\mu| \leq \frac{1}{2}$. If the first inequality holds, put $\mathbf{F}(n+1,\mu) = \mathbf{F}(n,\mu)$; otherwise let $\mathbf{F}(n+1,\mu)$ consist of $\mathbf{F}(n,\mu)$ and of $\mathbf{x}_{n+1}$. In either case we have $||\mathbf{F}(n+1,\mu)|-(n+1)\mu| \leq \frac{1}{2}$. Continuing in this way we obtain sets $\mathbf{F}(1,\mu), \mathbf{F}(2,\mu), \ldots$ which satisfy (58) for $n = 1, 2, \ldots$.

Let $\mathbf{G}_1(\mu)$ be the convex hull of the sets $\mathbf{F}(1,\mu), \mathbf{F}(2,\mu), \ldots$. Then $\mathbf{G}_1(\mu)$ is the convex hull of certain points among $\mathbf{x}_1, \mathbf{x}_2, \ldots$. If $\mu = 0$, the sets $\mathbf{F}(n,\mu)$ are empty, and hence so is $\mathbf{G}_1(\mu)$. If $0 < \mu \leq \frac{1}{3}$, put $n_0 = [1/(3\mu)]$ and apply (58) with $n = n_0$. Since $n_0\mu \leq \frac{1}{3}$, we obtain $|\mathbf{F}(n_0,\mu)| = 0$, and hence $\mathbf{x}_1,\ldots,\mathbf{x}_{n_0}$ do not lie in $\mathbf{G}_1(\mu)$. Thus $\mathbf{G}_1(\mu)$ is the convex hull of certain points among $\mathbf{x}_{n_0+1}, \mathbf{x}_{n_0+2}, \ldots$, and in view of (56) we obtain

$$\mu(\mathbf{G}_1(\mu)) \leq \frac{\pi}{4}(1/(4(n_0+1)))^2 \leq \left(\frac{\pi}{64}\right)(3\mu)^2 < \mu.$$

If $\frac{1}{3} < \mu \leq 1$, we have $\mu(\mathbf{G}_1(\mu)) \leq \pi/4 - 1/2 < 1/3 < \mu$. Hence always $\mu(\mathbf{G}_1(\mu)) \leq \mu$.

If $0 \leq \mu \leq \frac{1}{2}$, let $\mathbf{G}_2(\mu)$ be the convex hull of $\mathbf{F}(1,\mu), \mathbf{F}(2,\mu), \ldots$ and of the triangle $0 \leq x < 1, 0 \leq y < 1, y < x$. If $\mu = 1$, let $\mathbf{G}_2(\mu)$ be $\mathbf{U}^2$. In these cases we have $\mu(\mathbf{G}_2(\mu)) \geq \mu$. If $\frac{1}{2} < \mu < 1$, we have $\mathbf{x}_1 \in \mathbf{F}(1,\mu)$ by (57), and there is a smallest integer n_1 such that $\mathbf{x}_{n_1+1} \notin \mathbf{F}(n_1+1,\mu)$. In this case let $\mathbf{G}_2(\mu)$ be the convex hull of $\mathbf{F}(1,\mu), \mathbf{F}(2,\mu), \ldots$ and the open quadrilateral with vertices $(1,0), (1,1), \mathbf{x}_{n_1}, \mathbf{x}^*$, where $\mathbf{x}^* = (x^*, 1)$ is the intersection of the line $y = 1$ and the tangent to the circle $(x-1)^2 + y^2 = 1$ at $\mathbf{x}_{n_1}$. The quadrilateral contains the open rectangle with vertices $(x^*, y_{n_1}), (1, y_{n_1}), (x^*, 1), (1,1)$ of area $(1-x^*)\cdot(1-y_{n_1}) = ((1-y_{n_1})/(1-x_{n_1}))(1-y_{n_1}) > 1 - 2y_{n_1} > 1 - 2t_{n_1}$. Since $\mathbf{x}_1,\ldots,\mathbf{x}_{n_1}$ are in $\mathbf{F}(n_1+1,\mu)$, but $\mathbf{x}_{n_1+1}$ is not, (58) yields $|n_1 - (n_1+1)\mu| \leq \frac{1}{2}$, whence $n_1 \geq (\mu - \frac{1}{2})/(1-\mu)$, whence, by (55),

$$\mu(\mathbf{G}_2(\mu)) > 1 - 2t_{n_1} > 1 - (1/4n_1) \geqslant \begin{cases} 3/4 \geqslant \mu & \text{if} \quad 1/2 < \mu < 3/4, \\ 1 - (1-\mu) = \mu & \text{if} \quad 3/4 \leqslant \mu < 1. \end{cases}$$

Since $\mathbf{G}_1(\mu), \mathbf{G}_2(\mu)$ are convex sets with $\mu(\mathbf{G}_1(\mu)) \leq \mu \leq \mu(\mathbf{G}_2(\mu))$, there is a convex set $\mathbf{S}(\mu)$ with $\mathbf{G}_1(\mu) \subseteq \mathbf{S}(\mu) \subseteq \mathbf{G}_2(\mu)$ and with $\mu(\mathbf{S}(\mu)) = \mu$. The set $\mathbf{S}(\mu)$ lies in $\mathbf{U}^2$, since $\mathbf{G}_2(\mu)$ does. The intersection of $\mathbf{G}_1(\mu)$ with $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ is $\mathbf{F}(n,\mu)$, and the intersection of $\mathbf{G}_2(\mu)$ with $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ is $\mathbf{F}(n,\mu)$, so that also the intersection of $\mathbf{S}(\mu)$ with $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ is $\mathbf{F}(n,\mu)$. Therefore, by (58),

$$D(n, S(\mu)) = |Z(n, S(\mu)) - n\mu(S(\mu))| = ||F(n, \mu)| - n\mu| \leq \tfrac{1}{2}$$

for $n = 1, 2, \ldots$, so that $E(S(\mu)) \leq \frac{1}{2}$. Since μ was arbitrary in $0 \leq \mu \leq 1$, Theorem 2 is proved.

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Department of Mathematics, University of Colorado, Boulder, Colorado 80302